Integration of the mKdV hierarchy with integral type of source

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Abstract

We investigate the mKdV hierarchy with integral type of source (mKdVHWS), which consist of the reduced AKNS eigenvalue problem with $r = q$ and the mKdV hierarchy with extra term of the integration of square eigenfunction. First we propose a method to find the explicit evolution equation for eigenfunction of the auxiliary linear problems of the mKdVHWS. Then we determine the evolution equations of scattering data corresponding to the mKdVHWS which allow us to solve the equation in the mKdVHWS by inverse scattering transformation.

Keywords: soliton equation with integral type of source, Lax representation, the inverse scattering method.

1 Introduction

The soliton equations with integral type of source have important physical application, for example, the nonlinear Schrödinger equation with integral type of source (NLSEWS) is relevant to some problems of plasma physics and solid state physics [1]. It was shown in [2] that the NLSEWS can be integrated by the inverse scattering method for the Dirac operator. The key point of the application of the inverse scattering method to integration of the NLSEWS in [2] is the use of the determining relations playing the same role as different operator representations of the Lax type of nonlinear evolution equations integrable by various modifications of this method. Just using the determining relations Mel'nikov obtained the evolution equations for all the scattering data of the Dirac operator corresponding to NLSEWS. Similar method was used to investigate the KdV equation with integral type of source (KdVWS) in [3]. The reason for use of the determining relations in [2,3] is that the evolution equation of eigenfunction for eigenvalue problem corresponding to the NLSEWS and KdVWS was not found. In fact, establish of these determining relations and derivation of the evolution equations for all scattering data in [2,3] are quite complicated and required some skill.

In present paper we investigate the new mKdV hierarchy with integral type of sources (mKdVHWS), which consist of the reduced AKNS eigenvalue problem with $r = q$ and the mKdV hierarchy with extra term of the integration of square eigenfunction. We first present a method to construct the zero-curvature representation for mKdVHWS by finding the explicit evolution equation for eigenfunction of the auxiliary linear problrm for mKdVHWS. Then we present a way to determine the evolution equation for the scattering data corresponding to the mKdVHWS , which implies that the mKdVHWS can be integrated by the inverse scattering method. Comparing with the method by using determining relation in [2,3], the method proposed in this paper for determining the evolution equation of the scattering data is quite natural and simple. This general method can be applied to other (1+1)-dimensional soliton equations with integral type of source.

2 The mKdV hierarchy with integral type of source

Consider the reduced AKNS eigenvalue problem for $r = q$ [4]

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_x = U \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad U = \begin{pmatrix} -\lambda & q \\ q & \lambda \end{pmatrix}. \tag{2.1}$$

The adjoint representation of (2.1) reads [5]

$$V_x = [U, V] = UV - VU. \tag{2.2}$$

Set

$$V = \sum_{i=0}^{\infty} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{-i} \tag{2.3}$$

Eq. (2.2) yields

$$a_0 = -1, b_0 = c_0 = a_1 = 0, b_1 = c_1 = q$$
$$a_2 = \frac{1}{2}q^2, b_2 = -c_2 = -\frac{1}{2}q_x, \cdots,$$

and in general

$$b_{2m+1} = c_{2m+1} = Lb_{2m-1}, \quad b_{2m} = -c_{2m} = -\frac{1}{2}Db_{2m-1},$$

$$a_{2m+1} = 0, \quad a_{2m} = 2D^{-1}qb_{2m}. \tag{2.4}$$

where

$$L = \frac{1}{4}D^2 - qD^{-1}qD, \quad D = \frac{\partial}{\partial x}, \quad DD^{-1} = D^{-1}D = 1.$$

Set

$$V^{(2n+1)} = \sum_{i=0}^{2n+1} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{2n+1-i} \tag{2.5}$$

and take

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_{t_{2n+1}} = V^{(2n+1)} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{2.6}$$

Then the compatibility conditions of Eqs. (2.1) and (2.6) give rise to the mKdV hierarchy [4]

$$q_{t_{2n+1}} = -2b_{2n+2} = Db_{2n+1} = D\frac{\delta H_{2n+1}}{\delta q}, \quad n = 0, 1, \cdots, \tag{2.7}$$

where

$$H_{2n+1} = \frac{2a_{2n+2}}{2n+1}.$$

Using (2.1), we have

$$\frac{\delta \lambda}{\delta q} = \phi_1^2 - \phi_2^2, \qquad L(\phi_1^2 - \phi_2^2) = \lambda^2(\phi_1^2 - \phi_2^2). \tag{2.8}$$

The eigenvalue problem (2.1) with q vanishing rapidly as $|x|$ tends to infinity has not discrete eigenvalue. As proposed in [2,3,6,7], the mKdV hierarchy with integral type of source is defined by

$$q_{t_{2n+1}} = D[b_{2n+1} + \int_{-\infty}^{\infty} C(t,\zeta)(\phi_1^2(x,t,\zeta) - \phi_2^2(x,t,\zeta))d\zeta] \tag{2.9a}$$

$$\phi_{1,x} = -i\zeta\phi_1 + q\phi_2, \quad \phi_{2,x} = q\phi_1 + i\zeta\phi_2 \qquad \zeta \in (-\infty, \infty) \tag{2.9b}$$

we assume $q(x, t_{2n+1})$ tends rather quickly to zero as $x \to \pm\infty$. According to this condition we assume that

$$\phi_1(x,t,\zeta) \sim a(t,\zeta)exp(-i\zeta x), \qquad \phi_2(x,t,\zeta) \sim b(t,\zeta)exp(i\zeta x), \quad x \to -\infty \tag{2.10}$$

where $C = C(t,\zeta)$, a=$a(t,\zeta)$ and b=$b(t,\zeta)$ are complex functions of $t \geq 0$ and $\zeta \in (-\infty, \infty)$. Moreover we assume that the functions C, a and b are chosen so that the right-hand side of equation (2.9) determines the function absolutely integrable over x along the whole real axis. One can easily verify that the requirement will certainly be satisfied if the function E and Γ of the form as argued in [2]

$$E = |C(t,\zeta)|[|a(t,\zeta)| + |b(t,\zeta)|]^2$$

$$\Gamma = |\frac{\partial}{\partial \zeta}[C(t,\zeta)a^2(t,\zeta)]| + |\frac{\partial}{\partial \zeta}[C(t,\zeta)b^2(t,\zeta)]|$$

at any $t \geq 0$ satisfy the condition

$$\int_{-\infty}^{\infty}[E(t,\zeta) + \Gamma(t,\zeta) + \Gamma^2(t,\zeta)]d\zeta < \infty$$

3 The Lax representation

Following the method proposed in [6,7,8], in order to find the zero-curvature representation for (2.9), we first consider

$$D[b_{2n+1} + \int_{-\infty}^{\infty} C(t,\zeta)(\phi_1^2(x,t,\zeta) - \phi_2^2(x,t,\zeta))d\zeta] = 0 \tag{3.1a}$$

$$\phi_{1,x} = -i\zeta\phi_1 + q\phi_2, \quad \phi_{2,x} = q\phi_1 + i\zeta\phi_2 \qquad \zeta \in (-\infty,\infty) \tag{3.1b}$$

We can obtain the Lax representation for (3.1) by using the adjoint representation (2.2). According to (2.4), (2.8) and (3.1), we may define

$$\tilde{a}_i = a_i, \tilde{b}_i = b_i, \tilde{c}_i = c_i, \qquad i = 0, 1, \cdots, 2n,$$

$$\tilde{b}_{2n+2m+1} = \tilde{c}_{2n+2m+1} = L\tilde{b}_{2n+2m-1} = -\int_{-\infty}^{\infty} (i\zeta)^{2m+2} C(t,\zeta)[\phi_1^2(x,t,\zeta) - \phi_2^2(x,t,\zeta)]d\zeta$$

$$\tilde{b}_{2n+2m+2} = -\tilde{c}_{2n+2m+2} = -\frac{1}{2}D\tilde{b}_{2n+2m+1} = -\int_{-\infty}^{\infty} (i\zeta)^{2m+1} C(t,\zeta)[\phi_1^2(x,t,\zeta) + \phi_2^2(x,t,\zeta)]d\zeta$$

$$\tilde{a}_{2n+2m+2} = 2D^{-1}q\tilde{b}_{2n+2m+2} = \int_{-\infty}^{\infty} (i\zeta)^{2m+1} C(t,\zeta)\phi_1(x,t,\zeta)\phi_2(x,t,\zeta)d\zeta$$

$$\tilde{a}_{2n+2m+1} = 0, \qquad m = 0, 1, \cdots,$$

Then

$$N^{(2n+1)} = \begin{pmatrix} A^{(2n+1)} & B^{(2n+1)} \\ C^{(2n+1)} & D^{(2n+1)} \end{pmatrix} \equiv \lambda^{2n+1} \sum_{k=0}^{\infty} \begin{pmatrix} \tilde{a}_k & \tilde{b}_k \\ \tilde{c}_k & -\tilde{a}_k \end{pmatrix} \lambda^{-k} + \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}$$

where θ is some constant and

$$A^{(2n+1)} = \sum_{k=0}^{2n} a_k \lambda^{2n+1-k} + \theta + \int_{-\infty}^{\infty} \frac{2(i\zeta)(i\eta)C(t,\eta)\phi_1(x,t,\eta)\phi_2(x,t,\eta)}{(i\zeta)^2 - (i\eta)^2} d\eta$$

$$B^{(2n+1)} = \sum_{k=0}^{2n} b_k \lambda^{2n+1-k} + \int_{-\infty}^{\infty} \frac{i\zeta(i\zeta - i\eta)C(t,\eta)\phi_2^2(x,t,\eta) - i\zeta(i\zeta + i\eta)C(t,\eta)\phi_1^2(x,t,\eta)}{(i\zeta)^2 - (i\eta)^2} d\eta$$

$$C^{(2n+1)} = \sum_{k=0}^{2n} c_k \lambda^{2n+1-k} + \int_{-\infty}^{\infty} \frac{i\zeta(i\zeta + i\eta)C(t,\eta)\phi_2^2(x,t,\eta) - i\zeta(i\zeta - i\eta)C(t,\eta)\phi_1^2(x,t,\eta)}{(i\zeta)^2 - (i\eta)^2} d\eta$$

$$D^{(2n+1)} = -\sum_{k=0}^{2n} a_k \lambda^{2n+1-k} + \theta - \int_{-\infty}^{\infty} \frac{2(i\zeta)(i\eta)C(t,\eta)\phi_1(x,t,\eta)\phi_2(x,t,\eta)}{(i\zeta)^2 - (i\eta)^2} d\eta$$

also satisfies the adjoint representation (2.2), i.e.

$$N_x^{(2n+1)} = [U, N^{(2n+1)}], \tag{3.2}$$

which, in fact, gives rise to the Lax representation of (3.1). Since (3.1) is the stationary equation of (2.9), it is easy to find that the zero-curvature representation for the mKdV hierarchy with integral type of source (2.9) is given by

$$U_{t_{2n+1}} - N_x^{(2n+1)} + [U, N^{(2n+1)}] = 0, \tag{3.3}$$

with the auxiliary linear problems

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} -\lambda & q \\ q & \lambda \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} -i\zeta & q \\ q & i\zeta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{3.4a}$$

where $\lambda = i\zeta$ and

$$\psi_{1,t_{2n+1}}(x, t_{2n+1}, \zeta) = (A^{(2n+1)} + \theta)\psi_1 + B^{(2n+1)}\psi_2 \equiv \sum_{k=0}^{k=2n} (a_k\psi_1 + b_k\psi_2)\lambda^{2n+1-k} + \theta\psi_1$$

$$+ \int_{-\infty}^{\infty} \frac{i\zeta C(t,\eta)}{(i\zeta)^2 - (i\eta)^2} [2(i\eta)\phi_1(x,t,\eta)\phi_2(x,t,\eta)\psi_1 + (i\zeta - i\eta)\phi_2^2(x,t,\eta)\psi_2 - (i\zeta + i\eta)\phi_1^2(x,t,\eta)\psi_2] d\eta,$$

$$\psi_{2,t_{2n+1}}(x, t_{2n+1}, \zeta) = C^{(2n+1)}\psi_1 + (-A^{(2n+1)} + \theta)\psi_2 \equiv \sum_{k=0}^{k=2n} (c_k\psi_1 - a_k\psi_2)\lambda^{2n+1-k} + \theta\psi_2$$

$$+ \int_{-\infty}^{\infty} \frac{i\zeta C(t,\eta)}{(i\zeta)^2 - (i\eta)^2} [(i\zeta + i\eta)\phi_2^2(x,t,\eta)\psi_1 - (i\zeta - i\eta)\phi_1^2(x,t,\eta)\psi_1$$

$$-2(i\eta)\phi_1(x,t,\eta)\phi_2(x,t,\eta)\psi_2] d\eta. \tag{3.4b}$$

In this way we find the explicit evolution equations of eigenfunction ψ. Indeed, this kind of evolution equation of eigenfunction was not obtained in [2,3].

4 Evolution equation for the reflection coefficients

Now we can derive equations describing the evolution in time t of the S-matrix elements. This can be made as follows. We define the eigenfunctions $f^-(x,\zeta) = (f_1^-(x,\zeta), f_2^-(x,\zeta))^T$, $\bar{f}^-(x,\zeta) = (\bar{f}_1^-(x,\zeta), \bar{f}_2^-(x,\zeta))^T$, $f^+(x,\zeta) = (f_1^+(x,\zeta), f_2^+(x,\zeta))^T$ and $\bar{f}^+(x,\zeta) = (\bar{f}_1^+(x,\zeta), \bar{f}_2^+(x,\zeta))^T$ (here and hereafter the "T" means transposition) for the equation (3.4a), and the following asymptotics are fulfilled at any $\zeta \in (-\infty, \infty)$

$$f^-(x,\zeta) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\zeta x}, \quad \bar{f}^-(x,\zeta) \sim \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{i\zeta x}, as \ \ x \to -\infty \tag{4.1a}$$

$$f^+(x,\zeta)\sim\begin{pmatrix}0\\1\end{pmatrix}e^{i\zeta x},\quad \bar f^+(x,\zeta)\sim\begin{pmatrix}1\\0\end{pmatrix}e^{-i\zeta x}, as\ \ x\to+\infty \tag{4.1b}$$

As is known, the functions $f^-(x,\zeta)$ and $f^+(x,\zeta)$ admit an analytical continuation in the parameter ζ into the upper half-plane $\mathrm{Im}\zeta>0$, and the functions $\bar f^-(x,\zeta)$ and $\bar f^+(x,\zeta)$ admit an analytical continuation in the parameter ζ into the lower half-plane $\mathrm{Im}\zeta<0$. It is easily seen that at any real $\zeta\in(-\infty,\infty)$ the pair of functions $f^-(x,\zeta)$ and $\bar f^-(x,\zeta)$ forms a fundamental system of solutions to (3.4a). Hence, we may define

$$f^+(x,\zeta)=S_{12}(\zeta)\bar f^-(x,\zeta)+S_{22}(\zeta)f^-(x,\zeta) \tag{4.2a}$$

$$\bar f^+(x,\zeta)=S_{11}(\zeta)\bar f^-(x,\zeta)+S_{21}(\zeta)f^-(x,\zeta) \tag{4.2b}$$

where the quantities $S_{11}=S_{11}(\zeta)$, $S_{12}=S_{12}(\zeta)$, $S_{21}=S_{21}(\zeta)$ and $S_{22}=S_{22}(\zeta)$ are independent of x. Taking account of (4.1) and (4.2) we get that at any $\zeta\in(-\infty,\infty)$ the equality

$$S_{11}(\zeta)S_{22}(\zeta)-S_{12}(\zeta)S_{21}(\zeta)=1. \tag{4.3}$$

Under the assumption that $q(x,t)$ vanishes rapidly as $|x|\to\infty$, we have

$$a_0=-1,\ \ b_0=c_0=0,\ \ lim_{|x|\to\infty}a_j=lim_{|x|\to\infty}b_j=lim_{|x|\to\infty}c_j=0,\ \ j=1,2,\cdots,2n.$$

We denote the parameter θ in (3.4b) corresponding to $f^+(x,\zeta)$ by θ^+ and $\bar f^+(x,\zeta)$ by $\bar\theta^+$, respectively. Substituting $f^+(x,\zeta)$, $\bar f^+(x,\zeta)$ into (3.4b) respectively, we have

$$\begin{aligned}\frac{\partial f_1^+(x,\zeta)}{\partial t_{2n+1}}=&\{\sum_{k=0}^{2n}a_k(i\zeta)^{2n+1-k}+\theta^+-\oint_{-\infty}^{\infty}(\frac{\zeta}{\eta-\zeta}+\frac{\zeta}{\eta+\zeta})H(\eta)d\eta\\ &-\pi(i\zeta)H(\zeta)+\pi(i\zeta)H(-\zeta)\}f_1^+(x,\zeta)\\ &+\{\sum_{k=0}^{2n}(b_k(i\zeta)^{2n+1-k}+\oint_{-\infty}^{\infty}\frac{\zeta}{\eta-\zeta}H_1(\eta)d\eta+\oint_{-\infty}^{\infty}\frac{\zeta}{\eta+\zeta}H_2(\eta)d\eta\\ &+\pi(i\zeta)H_1(\zeta)-\pi(i\zeta)H_2(-\zeta)\}f_2^+(x,t,\zeta),\end{aligned} \tag{4.4a}$$

$$\begin{aligned}\frac{\partial f_2^+(x,\zeta)}{\partial t_{2n+1}}=&\{\sum_{k=0}^{2n}c_k(i\zeta)^{2n+1-k}-\oint_{-\infty}^{\infty}\frac{\zeta}{\eta-\zeta}H_2(\eta)d\eta-\oint_{-\infty}^{\infty}\frac{\zeta}{\eta+\zeta}H_1(\eta)d\eta\\ &-\pi(i\zeta)H_2(\zeta)+\pi(i\zeta)H_1(-\zeta)\}f_1^+(x,\zeta)\\ &+\{\sum_{k=0}^{2n}(-a_k(i\zeta)^{2n+1-k}-\oint_{-\infty}^{\infty}(\frac{i\zeta}{i\zeta-i\eta}-\frac{i\zeta}{i\zeta+i\eta})H(\eta)d\eta+\theta^+\end{aligned}$$

$$+\pi(i\zeta)H(\zeta)-\pi(i\zeta)H(-\zeta)\}f_2^+(x,\zeta), \tag{4.4b}$$

$$\frac{\partial \bar f_1^+(x,\zeta)}{\partial t_{2n+1}}=\{\sum_{k=0}^{2n}a_k(i\zeta)^{2n+1-k}+\bar\theta^+-\oint_{-\infty}^{\infty}(\frac{\zeta}{\eta-\zeta}+\frac{\zeta}{\eta+\zeta})H(\eta)d\eta$$
$$+\pi(i\zeta)H(\zeta)-\pi(i\zeta)H(-\zeta)\}\bar f_1^+(x,\zeta)$$
$$+\{\sum_{k=0}^{2n}(b_k(i\zeta)^{2n+1-k}+\oint_{-\infty}^{\infty}\frac{\zeta}{\eta-\zeta}H_1(\eta)d\eta+\oint_{-\infty}^{\infty}\frac{\zeta}{\eta+\zeta}H_2(\eta)d\eta$$
$$-\pi(i\zeta)H_1(\zeta)+\pi(i\zeta)H_2(-\zeta)\}\bar f_2^+(x,\zeta), \tag{4.4c}$$

$$\frac{\partial \bar f_2^+(x,\zeta)}{\partial t_{2n+1}}=\{\sum_{k=0}^{2n}c_k(i\zeta)^{2n+1-k}-\oint_{-\infty}^{\infty}\frac{\zeta}{\eta-\zeta}H_2(\eta)d\eta-\oint_{-\infty}^{\infty}\frac{\zeta}{\eta+\zeta}H_1(\eta)d\eta$$
$$+\pi(i\zeta)H_2(\zeta)-\pi(i\zeta)H_1(-\zeta)\}\bar f_1^+(x,\zeta)$$
$$+\{\sum_{k=0}^{2n}(-a_k(i\zeta)^{2n+1-k}-\oint_{-\infty}^{\infty}(\frac{i\zeta}{i\zeta-i\eta}-\frac{i\zeta}{i\zeta+i\eta})H(\eta)d\eta+\bar\theta^+$$
$$-\pi(i\zeta)H(\zeta)+\pi(i\zeta)H(-\zeta)\}\bar f_2^+(x,\zeta), \tag{4.4d}$$

where the integral $\oint$ is taken as the principal value, the quantities $\theta^+,\bar\theta^+$ will be determined in the next section, and

$$H(\eta)=C(t,\eta)\phi_1(x,t,\eta)\phi_2(x,t,\eta),$$

$$H_1(\eta)=C(t,\eta)\phi_1^2(x,t,\eta),\quad H_2(\eta)=C(t,\eta)\phi_2^2(x,t,\eta). \tag{4.5}$$

As $x\to-\infty$, we find that the following asymptotics are valid:

$$\oint_{-\infty}^{\infty}\frac{\zeta}{\eta-\zeta}H_1(\eta)d\eta\sim\pi(i\zeta)C(t,\zeta)a^2(\zeta,t)e^{-2i\zeta x},$$

$$\oint_{-\infty}^{\infty}\frac{\zeta}{\eta+\zeta}H_1(\eta)d\eta\sim\pi(i\zeta)C(t,-\zeta)a^2(-\zeta,t)e^{2i\zeta x},$$

$$\oint_{-\infty}^{\infty}\frac{\zeta}{\eta-\zeta}H_2(\eta)d\eta\sim-\pi(i\zeta)C(t,\zeta)b^2(\zeta,t)e^{2i\zeta x}, \tag{4.6}$$

$$\oint_{-\infty}^{\infty}\frac{\zeta}{\eta+\zeta}H_2(\eta)d\eta\sim-\pi(i\zeta)C(t,-\zeta)b^2(-\zeta,t)e^{-2i\zeta x},$$

Substituting (4.2) into (4.4) and using (4.6), as $x \to -\infty$, we have

$$\frac{\partial S_{22}(\zeta)}{\partial t_{2n+1}} = \{-(i\zeta)^{2n+1} + \theta^+ - \oint_{-\infty}^{\infty}(\frac{\zeta}{\eta-\zeta} + \frac{\zeta}{\eta+\zeta})h(\eta)d\eta$$
$$-\pi(i\zeta)h(\zeta) + \pi(i\zeta)h(-\zeta)\}S_{22}(\zeta)$$
$$-\{2\pi(i\zeta)h_1(\zeta) - 2\pi(i\zeta)h_2(-\zeta)\}S_{12}(\zeta),$$
$$\frac{\partial S_{12}(\zeta)}{\partial t_{2n+1}} = \{(i\zeta)^{2n+1} + \theta^+ + \oint_{-\infty}^{\infty}(\frac{\zeta}{\eta-\zeta} + \frac{\zeta}{\eta+\zeta})h(\eta)d\eta$$
$$+\pi(i\zeta)h(\zeta) - \pi(i\zeta)h(-\zeta)\}S_{12}(\zeta), \tag{4.7}$$
$$\frac{\partial S_{21}(\zeta)}{\partial t_{2n+1}} = \{-(i\zeta)^{2n+1} + \bar{\theta}^+ - \oint_{-\infty}^{\infty}(\frac{\zeta}{\eta-\zeta} + \frac{\zeta}{\eta+\zeta})h(\eta)d\eta$$
$$+\pi(i\zeta)h(\zeta) - \pi(i\zeta)h(-\zeta)\}S_{21}(\zeta),$$
$$\frac{\partial S_{11}(\zeta)}{\partial t_{2n+1}} = \{(i\zeta)^{2n+1} + \bar{\theta}^+ + \oint_{-\infty}^{\infty}(\frac{\zeta}{\eta-\zeta} + \frac{\zeta}{\eta+\zeta})h(\eta)d\eta$$
$$-\pi(i\zeta)h(\zeta) + \pi(i\zeta)h(-\zeta)\}S_{11}(\zeta)$$
$$-\{2\pi(i\zeta)h_2(\zeta) - 2\pi(i\zeta)h_1(-\zeta)\}S_{21}(\zeta),$$

where

$$h(\eta) = C(t,\eta)a(\eta,t)b(\eta,t)$$
$$h_1(\eta) = C(t,\eta)a^2(\eta,t), \quad h_2(\eta) = C(t,\eta)b^2(\eta,t)$$

One can easily see that if $C = 0$ or $a = b = 0$ then the resultant system (4.7) coincides with those equations which appear in the case of the mKdV hierarchy without a source. Also one can verify that system (4.7) is consistent with equality (4.3). Using (4.7), we find that the reflection coefficients

$$R_1(\zeta,t) = \frac{S_{11}(\zeta)}{S_{21}(\zeta)}, \quad R_2(\zeta,t) = \frac{S_{22}(\zeta)}{S_{12}(\zeta)} \tag{4.8}$$

satisfies the equation

$$\frac{\partial R_1(\zeta)}{\partial t_{2n+1}} = 2\{(i\zeta)^{2n+1} + \oint_{-\infty}^{\infty}(\frac{\zeta}{\eta-\zeta} + \frac{\zeta}{\eta+\zeta})h(\eta)d\eta$$
$$-\pi(i\zeta)h(\zeta) + \pi(i\zeta)h(-\zeta)\}R_1(\zeta)$$
$$-\{2\pi(i\zeta)h_2(\zeta) - 2\pi(i\zeta)h_1(-\zeta)\}, \tag{4.9a}$$

$$
\begin{aligned}
\frac{\partial R_2(\zeta)}{\partial t_{2n+1}} = 2\{&-(i\zeta)^{2n+1} - \oint_{-\infty}^{\infty}(\frac{\zeta}{\eta-\zeta} + \frac{\zeta}{\eta+\zeta})h(\eta)d\eta \\
&-\pi(i\zeta)h(\zeta) + \pi(i\zeta)h(-\zeta)\}R_2(\zeta) \\
&-\{2\pi(i\zeta)h_1(\zeta) - 2\pi(i\zeta)h_2(-\zeta)\}.
\end{aligned} \tag{4.9b}
$$

Then, it follows from (4.9) that the evolution of the reflection coefficients R_1, R_2 are influenced by the integral type of source which is integration of the square eigenfunctions belonging to the continuous spectrum of the spectral problem (2.1). For the case $r = q$, there is no discrete eigenvalue for the spectral problem (2.1) if the potential $q = q(x,t)$ tends rather quickly to zero as $|x| \to \infty$. The evolution equations for the reflection coefficients are presented by (4.9) which implies that the mKdV hierarchy with integral type of source can be solved by the inverse scattering method.

5 Conclusion

By means of the reduced AKNS eigenvalue problem with $r = q$ which has no discrete eigenvalue, we construct the mKdV hierarchy with integral type of source(mKdVHWS). We propose a method to find the evolution equation of eigenfunction corresponding to mKdVHWS and further to determine the evolution equation for scattering data which enable us to solve mKdVHWS by Inverse Scattering Transformation. Comparing with the method for determining the evolution equation for scattering data in [2,3], our approach is quite natural and simple.

It should be noted that the reduced AKNS spectral problem for $r = -q$ may have the discrete eigenvalue. In this case, the right-hand side of equation (2.9a) need to be added by the sum of square eigenfunctions of (2.9b) corresponding to the discrete eigenvalue. We will show in the further coming paper that the mKdV hierarchy with these two kinds of sources can also be integrated by Inverse Scattering Transformation.

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